CONDITIONAL MEASURES ON MV-ALGEBRAS

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Abstract In recent years many papers have been written generalizing some theorems, known from the Kolmogorovian probability theory, to MV-algebras. To achieve such results, so-called product MV-algebras were introduced and, using the product, the joint probability distribution was defined. In this paper we present an approach how to define the joint distributions on MV-algebras which are not necessarily closed under product. First we construct conditional measures on a given MV-algebra. And using these conditional measures we define the joint probability distributions.

1. PRELIMINARIES

Definition 1.1. *An MV-algebra is 5-tuple* $(\mathrm{M}, \oplus, *, \emptyset, 1)$ *such that* $(\mathrm{M}, \oplus, \emptyset)$ *is an Abelian monoid and moreover*

- $x^{**} = x$
- $\emptyset^* = 1$
- $x \oplus 1 = 1$
- $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$

Moreover for all $x, y \in \mathrm{M}$ we can define

$$x \otimes y = (x^* \oplus y^*)^*$$

$$x \wedge y = (x \oplus y^*) \otimes y$$

$$x \vee y = (x \otimes y^*) \oplus y$$

and then $(\mathrm{M}, \vee, \wedge, \emptyset, 1)$ is a bounded distributive lattice.

Example 1.1. ([2], [13]) Denote $(\Omega, \Im)$ a measurable space and μ some normed measure on that space. Then if we put M a system of $[0,1]$-valued $\Im$-measurable functions, closed under the following operations

$$f \oplus g(x) = \min(1, f(x) + g(x))$$

$$f \otimes g(x) = \max(0, f(x) + g(x) - 1)$$

$$f^*(x) = 1 - f(x),$$

and such that $0 \in M$, then M is an MV-algebra.

In this paper we will deal just with the MV-algebra M from Example 1.1. Denote $\nu(f) = \int f \, d\mu$, then ν is an additive measure on M, i.e.

$$\nu(f \oplus g) = \nu(f) + \nu(g)$$
(1)

if $f \le (1 - g)$

Definition 1.2. Events $f, g \in \mathrm{M}$ will be called ν-orthogonal (for short just orthogonal) if $\nu(f \wedge g) = 0$.

2. CONSTRUCTION OF CONDITIONAL MEASURES ON MV-ALGEBRAS

Definition 2.1. For any $g \in \mathrm{M}$ such that $\nu(g) > 0$ we say that $\gamma(.|g)$ is a conditional measure if and only if $\gamma(.|g)$ and $\gamma(.|g^*)$ are non-negative functions on M, bounded from above by 1, such that

A. $\nu(f) = \nu(g)\gamma(f|g) + \nu(g^*)\gamma(f|g^*)$, for any $f \in M$

B. $\gamma(f|h) = 0$, for all $f, h \in M$, which are orthogonal to each other

Definition 2.1. immediately implies the following

Lemma 2.1. For any $g \in \mathrm{M}$ such that $\nu(g) > 0$ there holds $\gamma(1|g) = 1$

Definition 2.2. Denote T the system of all transformations $\tau : M \to [0,1]^{\Omega}$ such that for each $f \in \mathrm{M}$

1. τ is $\Im$-measurable
2. $\int f \, d\mu = \int \tau(f) \, d\mu$
3. for any $x \in \Omega$ $f(x) = 0 \Leftrightarrow \tau(f)(x) = 0.$

Theorem 2.2. Let $\tau_2 \in T$ be such that for any $g \in M$ $\tau_2(g^*) = 1 - \tau_2(g)$ and $\tau_1 \in T$ be an arbitrary transformation. Define for any $f, g \notin M$

$$\gamma(f|g) = \begin{cases} 0 & \nu(g) = 0 \\ \nu(f) & \nu(g) = 1 \\ \dfrac{\int \tau_1(f)\tau_2(g)d\mu}{\int \tau_2(g)d\mu} & 0 < \nu(g) < 1 \end{cases} \qquad (2)$$

Then for any $g \in \mathrm{M}$ such that $\nu(g) > 0$, $\gamma(.|g)$ is a conditional measure.

Throughout this paper we will always denote by $\gamma(f|g)$ the measure given by Formula (2).

Definition 2.3. We will say that event f is independent of g with respect to a conditional measure γ if and only if $\nu(f) = \gamma(f|g)$.

In the sequel γ will always denote the conditional measure defined by Formula (2) from Theorem 2.2.

The independence of event f of g does not imply the independence of the event g of the event f. This non-symetric relation of independence allows us to distinguish between a cause and its effects. Similar results concerning the ortho-modular lattices have been achieved also by O. Nánásiová in [5]-[8].

Once having defined for any pair f, g of elements of the MV-algebra M the measure $\gamma(f|g)$, which is the conditional measure if $\nu(g) > 0$, we can define also the two-dimensional joint distribution on $\mathrm{M} \times \mathrm{M}$ - the measure (probability) of occurence of this pair f, g. This, in fact represents the interaction of f and g. And the interaction can be different if we change the order.

Definition 2.4. The measure of interaction of a pair $f, g \in \mathrm{M}$ will be denoted by $p(f, g)$ and defined as $p(f, g) = \gamma(f|g)\gamma(g|1)$.

If γ is defined as in Theorem 2.1 we get $p(f, g) = \int_{\Omega} \tau_1(f)\tau_2(g)\, d\mu$ where τ_1 and τ_2 are given transformations T such that $\tau_2(g^*) = 1 - \tau_2(g)$.

Theorem 2.3. Let p be a measure of interaction on the MV-algebra M and f, g be any elements of M. Then

$p(f,1) = p(1, f) = \nu(f)$

$p(f, g) = p(g, f) = 0$, if f and g are ν-orthogonal

$p(f, g) \le \min(\nu(f), \nu(g))$, particularly $p(f, f) \le \nu(f)$

the variables of p do not commute, i.e. in general $p(f, g) \ne p(g, f)$.

Example 2.1. Assume that $\Omega = [0,1]$ and μ is the Lebesgue measure. For any element $f \in \mathrm{M}$ put $\tau_2(f) = f$ and the transformation τ_1 will be defined by the following

$$\tau_1(f)(x) = \begin{cases} 1 & f(x) = 1 \\ 0 & f(x) = 0 \\ \dfrac{\int_A f(x)d\mu}{\mu(A)} & \textit{otherwise, where } A = \{x; 0 < f(x) < 1\}. \end{cases}$$

Let $f(x) = x$ and $g(x) = \max(0, x - 0.5)$. Then

	$x = 0$	$x = 1$	$0 < x < 1$
$(\tau_1(f))(x)$	0	1	0.5

	$x \le 0.5$	$x > 0.5$
$(\tau_1(g))(x)$	0	0.25

Then

$$p(f, g) = \int_0^1 0.5 g d\mu = 0.5 \int_{0.5}^1 (0.5 - x) d\mu = \frac{1}{16}.$$

$$p(g, f) = \int_{0.5}^1 0.25 x d\mu = \frac{3}{32}$$

$$p(f, f) = \int_0^1 0.5 x d\mu = \frac{3}{16}$$

$$p(g, g) = \int_{0.5}^1 0.25(x - 0.5) d\mu = \frac{1}{16}$$

3. SOME COMMENTS CONCERNING OBSERVABLES AND THEIR JOINT DISTRIBUTION

First we recall the definition of a tribe and of an observable.

Definition 3.1. An MV-algebra M will be called a tribe iff for any non-decreasing sequence of elements $\{f_i\}_{i=1}^{\infty}$ there holds $\bigvee_{i=1}^{\infty} f_i = f \in \mathrm{M}$.

From now on we will assume the MV-algebra to be a tribe.

Definition 3.2. An observable is a mappping λ from Borel sets $\mathrm{B}(R)$ into the MV-algebra M such that

$\lambda(R) = 1$.

If $A \cap B = \emptyset$, then $\lambda(A \cup B) = \lambda(A) \oplus \lambda(B)$ and $\lambda(A) \le \lambda(B^*)$.

If $A_n \quad A$, then $\lambda(A_n) \quad \lambda(A)$.

In a natural way for each observable λ we can define also its cummulative distributive function F_λ and its expectation $E(\lambda)$ by $F_\lambda(x) = \nu(\lambda((-\infty, x]))$

$$E(\lambda) = \int_{-\infty}^{\infty} x F_\lambda(dx) .$$

And, making a parallel to the measure of interaction p from Definition 2.4, we can define the joint probability distribution $P_{\lambda\kappa}$ for any pair of observables λ and κ by

$$P_{\lambda\kappa}(A,B) = p(\lambda(A), \kappa(B))$$

where A, B are Borel sets. This can be interpreted as the measure of interaction of the observables λ and κ. The basic properties of $P_{\lambda\kappa}$ can be just rewritten from Theorem 2.2.

It is also possible to define the mean interaction of the observables λ and κ, denoted by $C(\lambda, \kappa)$, as follows

$$C(\lambda, \kappa) = \int_{-\infty}^{\infty} (x - E(\lambda))(x - E(\kappa)) dF_{\lambda,\kappa}(x, x)$$

where $F_{\lambda,\kappa}(x, y) = P_{\lambda,\kappa}((-\infty, x], (-\infty, y])$ is the joint cummulative probability distribution. The investigation of the joint probability distributions (the interactions) of observables and of the corresponding mean interactions will be the topic of a next paper. Here we would like to point just to one very important property of the introduced notions, namely to the non-commutativity of the variables (observables) in the measure of interaction $F_{\lambda,\kappa}$ and in the mean interaction $C(\lambda, \kappa)$.

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